

THE TWO-PHASE FRACTIONAL OBSTACLE PROBLEM

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ABSTRACT. We study minimizers of the functional

$$\int_{B_1^+} |\nabla u|^2 x_n^a dx + 2 \int_{B_1'} (\lambda_+ u^+ + \lambda_- u^-) dx',$$

for $a \in (-1, 1)$. The problem arises in connection with heat flow with control on the boundary. It can also be seen as a non-local analogue of the, by now well studied, two-phase obstacle problem. Moreover, when u does not change signs this is equivalent to the fractional obstacle problem. Our main results are the optimal regularity of the minimizer, the separation of two free boundaries $\Gamma^+ = \partial'\{u(\cdot, 0) > 0\}$ and $\Gamma^- = \partial'\{u(\cdot, 0) < 0\}$ when $a \geq 0$, and the nonbranching of Γ^\pm when $a < 0$.

1. INTRODUCTION AND MOTIVATION

1.1. The problem. The main purpose of this paper is to study the minimizers of the energy functional

$$(1.1) \quad J_a(u) = \int_{D^+} |\nabla u|^2 x_n^a dx + 2 \int_{D'} (\lambda_+ u^+ + \lambda_- u^-) dx',$$

where D is a bounded domain in \mathbb{R}^n ,

$$D^+ := D \cap \{x_n > 0\}, \quad D' := D \cap \{x_n = 0\},$$

$a \in (-1, 1)$, $\lambda_\pm > 0$, and $u^\pm = \max\{\pm u, 0\}$. More precisely, we look at the functions u in the weighted Sobolev space $W^{1,2}(D^+, x_n^a)$, with prescribed boundary values

$$(1.2) \quad u = g \quad \text{on } (\partial D)^+ := \partial D \cap \{x_n > 0\},$$

in the sense of traces. This leads to a two-phase problem on D' if we identify the regions

$$\Omega^+ = \Omega_u^+ := \{u > 0\} \cap D', \quad \Omega^- = \Omega_u^- := \{u < 0\} \cap D'$$

as the phases of the minimizer u . We call their boundaries in D'

$$\Gamma^+ = \Gamma_u^+ := \partial'\Omega^+ \cap D', \quad \Gamma^- = \Gamma_u^- := \partial'\Omega^- \cap D'$$

free boundaries, since they are apriori unknown (∂' here stands for the boundary in \mathbb{R}^{n-1}). The free boundaries of this type are known as *thin free boundaries* in the literature, since they are formally of co-dimension two. Particular questions of interest are the regularity of the minimizers and the regularity and structure of their free boundaries.

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The problem above is strongly related to the so-called obstacle problem for fractional Laplacian $(-\Delta_{x'})^s$, $0 < s < 1$, or more precisely its localized version. The latter problem consists of minimizing the energy functional

$$(1.3) \quad E_a(v) = \int_{D^+} |\nabla v|^2 x_n^a dx, \quad a = 1 - 2s$$

among all functions $v \in W^{1,2}(D^+, x_n^a)$ satisfying

$$(1.4) \quad v = h \quad \text{on } (\partial D)^+,$$

$$(1.5) \quad v \geq \phi \quad \text{on } D'.$$

The function ϕ above is known as the *thin obstacle*. In the particular case of $a = 0$, this problem is the scalar version of the Signorini problem, also known as the *thin* (or *boundary*) *obstacle problem*. The free boundary in this problem is

$$\Gamma_v = \partial' \{v(\cdot, 0) > \phi\} \cap D'.$$

There has been a surge of interest in such problems in recent years, because of the development of new tools such as the Caffarelli-Silvestre extension, Almgren-type monotonicity formulas, etc, that allowed significant advances in the study of the regularity properties of the free boundaries; see [AC04, Sil07, ACS08, CSS08, GP09].

The exact connection between the minimizers of (1.1)–(1.2) and (1.3)–(1.5) is given in the following (relatively straightforward) proposition.

Proposition 1.1. *Let u be a minimizer of (1.1)–(1.2) such that $u \geq 0$ on D' . Then $v(x) = u(x) - \frac{\lambda_+}{1-a} x_n^{1-a}$ is a minimizer of (1.3)–(1.5) with $h(x) = g(x) - \frac{\lambda_+}{1-a} x_n^{1-a}$ and $\phi = 0$.*

Proof. The proof is an immediate corollary of the following computation:

$$\begin{aligned} E_a(v) &= \int_{D^+} |\nabla u - \lambda_+ x_n^{-a} e_n|^2 x_n^a \\ &= \int_{D^+} (|\nabla u|^2 x_n^a + \lambda_+^2 x_n^{-a} - 2\lambda_+ \partial_{x_{n-1}} u) \\ &= \int_{D^+} |\nabla u|^2 x_n^a + 2 \int_{D'} (\lambda_+ u^+) + C(D^+, g) \\ &= J_a(u) + C(D^+, g). \end{aligned} \quad \square$$

In particular, when $u \geq 0$ we see that $u(x', 0) \equiv v(x', 0)$ and consequently

$$\Gamma_u^+ = \Gamma_v,$$

reducing the study of the free boundary in (1.1)–(1.2) to that in the fractional obstacle problem (1.3)–(1.5). For the reasons above, we call the minimization problem (1.1)–(1.2) the *two-phase fractional obstacle problem*. When $a = 0$ we also call it the *two-phase thin obstacle problem*.

The two-phase fractional obstacle problem can also be viewed as a nonlocal version of the so-called *two-phase obstacle problem* (also known as the two-phase membrane problem)

$$(1.6) \quad \Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \quad \text{in } D.$$

Solutions of (1.6) can be obtained by minimizing the the energy functional

$$\tilde{J}(u) = \int_D (|\nabla u|^2 + 2\lambda_+ u^+ + 2\lambda_- u^-) dx.$$

This problem has been studied in detail in a series of papers [Wei01a, Ura01, SUW04, SW06, SUW07]. In particular, it is known that the minimizers are $C^{1,1}$ -regular and that the free boundaries $\tilde{\Gamma}^\pm := \partial\{u > 0\} \cap D$ are graphs of C^1 functions near two-phase points (i.e. points on $\tilde{\Gamma}^+ \cap \tilde{\Gamma}^-$). Away from two-phase points, the problem is locally equivalent to the classical obstacle problem, which is extensively studied in the literature. We also refer to the book [PSU12] for the introduction to this and other obstacle-type problems.

1.2. Main results. Our first result concerns the regularity of the minimizers.

Theorem 1.2 (Optimal regularity). *Let u be a minimizer of (1.1). Then for any $K \Subset D^+ \cup D'$ we have*

$$\begin{aligned} u &\in C^{0,1-a}(K), \quad \text{if } a \geq 0 \\ u &\in C^{1,-a}(K), \quad \text{if } a < 0 \end{aligned}$$

with bounds on their respective norms, depending only on K , D , a , λ_\pm , n , and $\|u\|_{L^2(D^+)}$.

Note that this regularity is sharp, because of the explicit example

$$u(x) = \frac{\lambda_+}{(1-a)} x_n^{1-a},$$

see Proposition 1.1.

Our second result is about the structure of the free boundary, which effectively reduces the problem to the (one-phase) fractional obstacle problem.

Theorem 1.3 (Structure of the free boundary). *Let u be a minimizer of (1.1).*

(i) *If $a \geq 0$, then*

$$\Gamma^+ \cap \Gamma^- = \emptyset.$$

(ii) *If $a < 0$, then*

$$|\nabla_{x'} u| \neq 0 \quad \text{on } \Gamma^+ \cap \Gamma^-.$$

In particular, by the implicit function theorem, $\Gamma^+ \cap \Gamma^-$ is locally a $C^{1,-a}$ surface of dimension $(n-2)$.

Thus, by Proposition 1.1, the study of the (remaining portion of the) free boundary is reduced to the fractional obstacle problem.

1.3. Structure of the paper.

- In Section 2, we describe the notation used throughout the paper.
- Section 3 contains a couple of applications that motivated our study.
- In Section 4 we collected some results on a -harmonic functions: interior estimates, Liouville-type theorem and Almgren's frequency formula, as well as some preliminary results on our problem: energy inequality, local boundedness, Weiss-type monotonicity formula (the proof of the latter is given in Appendix). It also contains the Alt-Caffarelli-Friedman monotonicity formula to be used in the case $a = 0$.
- In Section 5 we prove some preliminary regularity on the minimizers: namely $C^{0,(1-a)/2}$, when $a \geq 0$, and $C_{x'}^{1,\alpha}$ for $\alpha < -a$, when $a < 0$.

- Section 6 contains the proof of Theorem 1.2. We first show the optimal growth rate of the minimizers from the free boundary (Theorem 6.1) which then implies the optimal regularity (Theorem 6.2).
- In Section 7 we prove the nondegeneracy of each phase near their respective free boundaries.
- In Section 8 we study the blowups, by showing first that they are homogeneous of degree $(1 - a)$ and then classifying all such minimizers.
- Finally, in Section 9 we prove Theorem 1.3. In fact, we prove its stronger version (Theorem 9.1) that the free boundaries Γ^+ and Γ^- (or their portions where $|\nabla_{x'} u| = 0$ if $a < 0$) are uniformly separated. We conclude the paper with an example of a solution when $a < 0$ where Γ^+ and Γ^- have a common part with nonvanishing thin gradient $|\nabla_{x'} u|$.

2. NOTATION

Throughout the paper we will be using the following notation:

- We denote a point $x \in \mathbb{R}^n$ by (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Moreover, in certain cases we will identify $x' \in \mathbb{R}^{n-1}$ with $(x', 0) \in \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$. We will refer to $\mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\}$ as the *thin space*.
- $\nabla u = (\partial_{x_1} u, \dots, \partial_{x_{n-1}} u, \partial_{x_n} u)$: the full gradient.
- $\nabla_{x'} u = (\partial_{x_1} u, \dots, \partial_{x_{n-1}} u)$: the gradient in the thin space, or the thin gradient.
- u_{x_i} : alternative notation for the partial derivative $\partial_{x_i} u$, $i = 1, \dots, n$.
- For a set $E \subset \mathbb{R}^n$ we define

$$E' := E \cap (\mathbb{R}^{n-1} \times \{0\})$$

- For $E \subset \mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\}$, $\partial' E$ is the boundary in the topology of \mathbb{R}^{n-1} .
- For a function u , we define $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, the positive and negative parts of u so that, $u = u^+ - u^-$.
- We call Γ^+ and Γ^- the free boundaries where

$$\Gamma^+ = \partial' \{u(\cdot, 0) > 0\} \text{ and } \Gamma^- = \partial' \{u(\cdot, 0) < 0\}$$

- We will for $x_0 \in \Gamma^+ \cup \Gamma^-$, denote by u_{r, x_0} , the rescaled function

$$\frac{u(rx + x_0)}{r^{1-a}}.$$

If there is a subsequence u_{r_j, x_0} , where $r_j \rightarrow 0$, converging locally uniformly to a function u_{0, x_0} we will say that u_{0, x_0} is a *blow-up* of u at the point x_0 .

- $L^2(D^+, x_n^a)$ [respectively $L^2(D, |x_n|^a)$], for $|a| < 1$, is the weighted Lebesgue space with the square of the norm given by

$$\int_{D^+} |f(x)|^2 x_n^a dx \quad \left[\text{respectively } \int_D |f(x)|^2 |x_n|^a dx \right].$$

- $W^{1,2}(D^+, x_n^a)$ [respectively $W^{1,2}(D, |x_n|^a)$], for $|a| < 1$, is the weighted Sobolev space of functions $u \in L^2(D^+, x_n^a)$ [respectively $L^2(D, |x_n|^a)$] such that the distributional derivatives $\partial_{x_i} u$, $i = 1, \dots, n$, are also in $L^2(D^+, x_n^a)$ [respectively $L^2(D, |x_n|^a)$]. For some of the basic results for these spaces such as the Sobolev embedding and Poincaré inequalities, we refer to [FKS82]. See also [Kil97].

It is known that if there is a bi-Lipschitz transformation that will take (D, D') to (B_1, B'_1) then functions in $W^{1,2}(D^+, x_n^a)$ [or $W^{1,2}(D, |x_n|^a)$] have traces in $L^2(D')$ and $L^2((\partial D)^+, x_n^a)$ [or $L^2(\partial D, |x_n|^a)$]. Furthermore, the corresponding

trace operators are compact. See [Nek93] for the first trace operator. Compactness of the second one follows from the approximation by Lipschitz functions as indicated in Remark 6 in [Kil97], similarly to the proof of Theorem 8 there.

- We say that $u \in W^{1,2}(D, |x_n|^a)$ is *a-harmonic* in an open set $D \subset \mathbb{R}^n$ if it is a weak solution of

$$L_a u := \operatorname{div}(|x_n|^a \nabla u) = 0.$$

Equivalently, *a-harmonic* functions are the minimizers of the weighted Dirichlet integral

$$E_a(u) := \int_D |\nabla u|^2 |x_n|^a \, dx,$$

among all functions with the same trace on ∂D .

- For $u \in W^{1,2}(D, |x_n|^a)$, by an *a-harmonic replacement* of u on $U \Subset D$ we understand a function v coinciding with u on $D \setminus U$, *a-harmonic* in U , with the same boundary values on ∂U as u in the sense $v - u \in W_0^{1,2}(U, |x_n|^a)$.
- Very often we will use the following convention for integrals. If the measure of integration is not indicated then it is either the standard Lebesgue measure dx in \mathbb{R}^n , or $(n-1)$ -dimensional Hausdorff measure $d\mathcal{H}^{n-1}$ (also denoted dx' when restricted to \mathbb{R}^{n-1}).

3. MOTIVATION

In this section we describe some applications that motivated us to study the problem (1.1)–(1.2), in addition to those indicated in the introduction.

3.1. Temperature control problems. One of the applications of the two-phase obstacle problem (1.6) is the temperature control problem through the interior, see Duvaut-Lions [DL76, Chapter I, §2.3.2]. More specifically, suppose that we want to maintain the temperature $u(x)$ in the domain D close to the range (θ_-, θ_+) , where $\theta_-(x) \leq \theta_+(x)$ are two given functions in D . For that purpose we have an array of cooling/heating devices that are distributed evenly over the domain D . The devices are assumed to be of limited power, with their generated heat flux $-f$ in the range $[-\lambda_-, \lambda_+]$, $\lambda_{\pm} > 0$. The device will generate no heat if $u(x) \in [\theta_-(x), \theta_+(x)]$, however, outside that range the corrective heat flux F will be injected according to the following rule:

$$F = F(u) = \begin{cases} \min\{k_+(u - \theta_+), \lambda_+\}, & u > \theta_+, \\ \max\{k_-(u - \theta_-), -\lambda_-\}, & u < \theta_-, \end{cases}$$

where $k_{\pm} \geq 0$. In the equilibrium state, the temperature distribution will satisfy

$$\Delta u = F(u) \quad \text{in } D.$$

Assuming $\theta_{\pm} = 0$ and $k_{\pm} = +\infty$, the equation becomes

$$\Delta u = \lambda_+ \chi_{\{u > 0\}} - \lambda_- \chi_{\{u < 0\}} \quad \text{in } D,$$

which is nothing but the two-phase obstacle problem.

Suppose now that we want to regulate the temperature in $D^+ = D \cap \{x_n > 0\}$ with the heating/cooling devices that are evenly distributed along the “wall” $D' = D \cap \{x_n = 0\}$. Then in the equilibrium position, the temperature will satisfy

$$\Delta u = 0 \quad \text{in } D^+, \quad \partial_{x_n} u = F(u) \quad \text{on } D'.$$

In the limiting case $\theta_{\pm} = 0$ and $k_{\pm} = \infty$ this transforms into the variational inequality

$$\Delta u = 0 \quad \text{in } D, \quad \partial_{x_n} u \in \bar{F}(u) \quad \text{on } D',$$

where \bar{F} is the maximal monotone graph

$$(3.1) \quad \bar{F}(u) = \begin{cases} \{\lambda_+\}, & u > 0, \\ [-\lambda_-, \lambda_+], & u = 0, \\ \{-\lambda_-\}, & u < 0. \end{cases}$$

Equivalently, the solutions of this variational inequality can be obtained by minimizing the energy functional

$$J_0(u) = \int_{D^+} |\nabla u|^2 dx + 2 \int_{D'} (\lambda_+ u^+ + \lambda_- u^-) dx',$$

subject to a Dirichlet boundary condition on $(\partial D)^+$ (cf. Lemma 4.1). This is a particular case ($a = 0$) of the problem that we intend to study in this paper.

3.2. Two-phase problem for fractional Laplacian. Suppose now that Σ is a bounded open set in \mathbb{R}^{n-1} , $0 < s < 1$, and we consider the minimizers of the energy functional

$$j_s(v) = c_{n,s} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{(v(x') - v(y'))^2}{|x' - y'|^{n-1+2s}} + 2 \int_{\mathbb{R}^{n-1}} (\lambda_+ u^+ + \lambda_- u^-),$$

among all function such that $v = g$ on $\mathbb{R}^{n-1} \setminus \Sigma$. Here $c_{n,s} > 0$ is a normalization constant. Then it is easy to see that the minimizers will satisfy the variational inequality

$$(3.2) \quad -(-\Delta_{x'})^s v \in \bar{F}(v) \quad \text{on } \Sigma,$$

with the maximal monotone graph \bar{F} in (3.1). To see the connection with the two-phase thin obstacle problem, we use the Caffarelli-Silvestre extension [CS07] for v . Namely, let u be a function \mathbb{R}_+^n that satisfies

$$\begin{aligned} L_a u &:= \operatorname{div}(x_n^a \nabla u) = 0 \quad \text{in } \mathbb{R}_+^n \\ u(x', 0) &= v(x') \quad \text{for } x' \in \mathbb{R}^{n-1}, \end{aligned}$$

where $a = 1 - 2s \in (-1, 1)$. (The initial condition is understood in the boundary trace sense.) Then we can recover the fractional Laplacian of v on \mathbb{R}^{n-1} by

$$(-\Delta_{x'})^s v(x') = C_{n,s} \lim_{x_n \rightarrow 0^+} x_n^a \partial_{x_n} u(x', x_n),$$

in the weak sense. Suppose now that there is a smooth domain D in \mathbb{R}^n is such that $\Sigma = D' = D \cap \{x_n = 0\}$. Then, using this extension we see that the same u can be recovered by minimizing the energy functional J_a among all function u with fixed Dirichlet condition on $(\partial D)^+$, which is exactly the problem we stated in the beginning of the introduction.

4. PRELIMINARIES

In this section we present certain results about the minimizers of (1.1) and a -harmonic functions that we will utilize later on.

We start with a remark that the results that we are interested in are local in nature and therefore without loss of generality we may assume that $D = B_1$ in (1.1). Moreover, it will also be convenient to extend the minimizers u from B_1^+ to the full ball B_1 by even symmetry

$$u(x', -x_n) = u(x', x_n).$$

Then such u is in $W^{1,2}(B_1, |x_n|^a)$ and is the minimizer of the functional

$$(4.1) \quad J_a(u) = \frac{1}{2} \int_{B_1} |\nabla u|^2 |x_n|^a dx + 2 \int_{B_1'} (\lambda_+ u^+ + \lambda_- u^-) dx'$$

subject to the boundary condition

$$(4.2) \quad u = g \quad \text{on } \partial B_1,$$

in the sense of traces, where $g \in L^2(\partial B_1, |x_n|^a)$ is also evenly extended to ∂B_1 from $(\partial B_1)^+$.

The existence of minimizers of (4.1)–(4.2), follows by the direct method in the calculus of variation, since J_a is continuous on $W^{1,2}(B_1, |x_n|^a)$. The uniqueness follows from the strict convexity of J_a on the closed convex subset of functions in $W^{1,2}(B_1, |x_n|^a)$ satisfying (4.2). The next lemma identifies the variational inequality satisfied by the minimizers.

Lemma 4.1 (Variational inequality). *Let u be a minimizer of (4.1). Then for $x' \in B_1'$ we have*

$$(4.3) \quad - \lim_{x_n \rightarrow 0^+} x_n^a u_{x_n}(x', x_n) \in \bar{F}(u) = \begin{cases} \{\lambda_+\} & u > 0 \\ [-\lambda_-, \lambda_+] & u = 0 \\ \{-\lambda_-\} & u < 0 \end{cases}$$

in the sense that there exists a measurable function f in B_1' such that

$$(4.4) \quad \begin{aligned} f(x') &\in \bar{F}(u(x', 0)) \quad \text{for a.e. } x' \in B_1', \text{ and} \\ - \int_{B_1^+} |x_n|^a \nabla u \nabla \psi dx &= \int_{B_1'} f \psi dx' \quad \text{for any } \psi \in C_0^\infty(B_1). \end{aligned}$$

Remark 4.2. Taking $\psi(x', x_n) = \gamma(x')\eta(x_n)$ in (4.4) with $\gamma \in C_0^\infty(B_1')$ and $\eta \in C_0^\infty([0, 1))$, $\eta \equiv 1$ near 0, we obtain that the convergence in (4.3) can be understood also in the sense of distributions

$$(4.5) \quad \lim_{x_n \rightarrow 0^+} \int_{B_1'} x_n^a u_{x_n}(x', x_n) \gamma(x') dx' = \int_{B_1'} f \gamma dx'.$$

Proof. In fact, we will prove something more general: if u is a minimizer of the functional

$$J(u) = \int_{B_1^+} |\nabla u|^2 x_n^a + 2 \int_{B_1'} \Psi(u),$$

with a fixed Dirichlet data on $(\partial B_1)^+$, where $\Psi(s)$ is a convex function of $s \in \mathbb{R}$, then

$$\int_{B_1'} \Psi'(u-) \psi \leq - \int_{B_1^+} |x_n|^a \nabla u \nabla \psi \leq \int_{B_1'} \Psi'(u+) \psi$$

for any nonnegative $\psi \in C_0^\infty(B_1)$. Indeed, for such ψ consider a competing function $u + \epsilon\psi$. From the minimality of u we then have

$$0 \leq \frac{J(u + \epsilon\psi) - J(u)}{\epsilon} = 2 \int_{B_1^+} x_n^a \nabla u \nabla \psi + \epsilon \int_{B_1^+} |\nabla \psi|^2 x_n^a + \int_{B_1'} \frac{\Psi(u + \epsilon\psi) - \Psi(u)}{\epsilon}.$$

Now, noting that $(\Psi(u + \epsilon\psi) - \Psi(u))/\epsilon$ is monotone in $\epsilon > 0$ and converges to $\Psi'(u+)\psi$, by the monotone convergence theorem we will arrive at

$$- \int_{B_1^+} |x_n|^a \nabla u \nabla \psi \leq \int_{B_1'} \Psi'(u+)\psi.$$

The inequality from below is proved similarly. \square

The minimizers of (4.1) also enjoy the following comparison principle.

Lemma 4.3 (Comparison principle). *Let u, v be two minimizers of the functional (4.1) with $u|_{\partial B_1} \leq v|_{\partial B_1}$. Then $u \leq v$ in B_1 .*

Proof. If we define $\bar{w} := \max\{u, v\}$ and $\underline{w} := \min\{u, v\}$, it is straightforward to verify that

$$J(\bar{w}) + J(\underline{w}) = J(u) + J(v)$$

Since the functional is strictly convex on functions with the same boundary data, minimizers are unique. Since $\bar{w}|_{\partial B_1} = v|_{\partial B_1}$ and $\underline{w}|_{\partial B_1} = u|_{\partial B_1}$, we may therefore conclude that $u = \underline{w}$, $v = \bar{w}$, readily implying that $u \leq v$ in B_1 . \square

Corollary 4.4. *If the boundary data are symmetric about the line $(0, \dots, 0, x_n)$, then the minimizer is symmetric about the line $(0, \dots, 0, x_n)$.*

Proof. Any rotation will be a minimizer, and minimizers are unique. \square

Lemma 4.5. *Let u be a minimizer of (4.1). Then u^\pm are a -subharmonic functions, i.e., $L_a u^\pm \geq 0$ in the weak sense*

$$\int_{B_1} \nabla(u^\pm) \nabla \psi |x_n|^a \leq 0$$

for any nonnegative $\psi \in W_0^{1,2}(B_1, |x_n|^a)$.

Proof. For the nonnegative test function $\psi \in W_0^{1,2}(B_1, |x_n|^a)$ and small $\epsilon > 0$, let

$$u_\epsilon = (u - \epsilon\psi)^+ - u_-.$$

Since $u = u_\epsilon$ on ∂B_1 , from minimality of u we must have $J_a(u) \leq J_a(u_\epsilon)$. Noticing that, $u_\epsilon^+ = (u - \epsilon\psi)^+ \leq u^+$ and $u_\epsilon^- = u^-$ we therefore have

$$\begin{aligned} \int_{B_1 \cap \{u > 0\}} |\nabla u|^2 |x_n|^a &\leq \int_{B_1 \cap \{u > \epsilon\psi\}} |\nabla u - \epsilon \nabla \psi|^2 |x_n|^a \\ &\leq \int_{B_1 \cap \{u > 0\}} |\nabla u - \epsilon \nabla \psi|^2 |x_n|^a. \end{aligned}$$

This readily implies that

$$\int_{B_1} \nabla(u^+) \nabla \psi |x_n|^a = \int_{B_1 \cap \{u > 0\}} \nabla u \nabla \psi |x_n|^a \leq 0.$$

This proves a -subharmonicity of u^+ . Arguing similarly, we establish the same fact also for u^- . \square

Corollary 4.6 (Energy inequality). *Let u be a minimizer of (4.1). Then we have the following inequality*

$$\int_{B_r} |\nabla u|^2 |x_n|^a \leq \frac{C}{r^2} \int_{B_{2r}} u^2 |x_n|^a.$$

Proof. Since u^\pm are nonnegative and a -subharmonic, the standard proof of the Caccioppoli inequality (with test functions $u^\pm \eta^2$) applies and gives

$$\int_{B_r} |\nabla u_\pm|^2 |x_n|^a \leq \frac{C_n}{r^2} \int_{B_{2r}} u_\pm^2 |x_n|^a.$$

Combining these two inequalities, we complete the proof. \square

Corollary 4.7 (Local boundedness). *Let u be a minimizer (4.1). Then*

$$\|u\|_{L^\infty(B_{1/2})} \leq C(n, a) \|u\|_{L^2(B_1, |x_n|^a)}$$

Proof. This follows from L^∞ - L^2 estimates for a -subharmonic functions u^\pm , see [FKS82]. \square

For our further study, we will need some properties of a -harmonic functions.

We start with the observation that Corollaries 4.6 and 4.7 are applicable also to a -harmonic functions (e.g., take $\lambda_\pm = 0$ in the proofs above). Besides, since $|x_n|^a$ is an A_2 weight, we have the following result from [FKS82].

Lemma 4.8 (Hölder continuity). *Let u be a -harmonic in B_1 . Then for some $\alpha > 0$, $u \in C^{0,\alpha}(B_{1/2})$ and*

$$\|u\|_{C^{0,\alpha}(B_{1/2})} \leq C(n, a, \alpha) \|u\|_{L^2(B_1)}. \quad \square$$

Next, we the following derivative estimates in x' directions (see [CSS08, Corollary 2.5]).

Lemma 4.9 (Interior estimates). *Let u be a -harmonic in a ball $B_r(x)$. Then for any positive integer k*

$$\sup_{B_{r/2}(x)} |D_{x'}^k u| \leq \frac{C(n, a, k)}{r^k} \operatorname{osc}_{B_r(x)} u \quad \square$$

Note that because of the weight $|x_n|^a$, similar estimates for the derivatives in x_n must be weighted accordingly.

Further, at several points we will need the following result [CSS08, Lemma 2.7].

Lemma 4.10 (Liouville-type theorem). *Let u be a -harmonic in \mathbb{R}^n , even with respect to x_n and*

$$|u(x)| \leq C(1 + |x|^k),$$

for some $C, k > 0$. Then u is a polynomial. \square

We next state an Almgren-type frequency formula, which first appeared in [CS07, Theorem 6.1]:

Lemma 4.11 (Almgren-type frequency formula). *Let u be a -harmonic in B_1 . Then the function*

$$N(r) = N(u, r) = \frac{r \int_{B_r} |\nabla u|^2 |x_n|^a}{\int_{\partial B_r} u^2 |x_n|^a}$$

is increasing in r , for $r \in (0, 1)$. Moreover, $N(r)$ is constant in (r_1, r_2) if and only if u is homogeneous of degree $N(r_1)$ in $B_{r_2} \setminus B_{r_1}$. \square

Following the proof of [Wei01b, Lemma 4.1] and using Lemma 4.10 and Lemma 4.11 we have the corollary below:

Corollary 4.12. *Let u be a -harmonic in B_1 , even in x_n , with $u(0) = 0$. If $a < 0$ assume also that $|\nabla_{x'} u(0)| = 0$. Then*

$$\int_{B_r} |\nabla u|^2 |x_n|^a \geq C(a) \int_{\partial B_r} u^2 |x_n|^a,$$

where $C(a) = 1$ if $a \geq 0$ and $C(a) = 2$ if $a < 0$. Moreover, equality holds if and only if u is homogeneous of degree $C(a)$.

Proof. We first prove the inequality by contradiction. If the statement is false then there is $r_0 \in (0, 1)$ such that $N(r_0) < C(a)$. By Lemma 4.11, this implies $N(r) < C(a)$ for all $r \in (0, r_0)$. Let

$$w_r(x) = \frac{w(rx)}{\left(\int_{\partial B_1} w^2(rx) |x_n|^a \right)^{\frac{1}{2}}}.$$

Then

$$(4.6) \quad \int_{\partial B_1} w_r^2 |x_n|^a = 1,$$

w_r is bounded in $W^{1,2}(B_1, |x_n|^a)$ and a -harmonic in B_1 . By the compactness of the trace operator, Lemma 4.8 and Corollary 4.6, we can find a sequence $r_j \rightarrow 0+$ such that w_{r_j} converges strongly in $L^2(\partial B_1, |x_n|^a) \cap W^{1,2}(B_{1/2}, |x_n|^a) \cap C(B_{1/2})$ (and also in $C_{x'}^1(B_{1/2})$ if $a < 0$ from Lemma 4.9) to a limit function w_0 . Hence, for $s \in (0, 1/2)$ there holds

$$N(0+, w) = \lim_{r_j \rightarrow 0} N(sr_j, w) = \lim_{r_j \rightarrow 0} N(s, w_{r_j}) = N(s, w_0),$$

so that $N(s, w_0)$ is constant. Therefore, by Lemma 4.11, w_0 is homogeneous of degree $N(0+, w) < C(a)$ in $B_{1/2}$. We can extend w_0 to a -homogeneous function in whole \mathbb{R}^n . Note that a -harmonic functions are real analytic off the thin space $\mathbb{R}^{n-1} \times \{0\}$, and therefore a -harmonic continuations are unique.

Now, if $a \geq 0$, Lemma 4.10 implies that $w_0 = 0$ contradicting (4.6).

If $a < 0$, then Lemma 4.10 implies that w_0 is linear. Due to the extra assumption when $a < 0$, $w_0 = 0$ and a contradiction is reached again.

Finally, if equality holds for some $r_0 \in (0, 1)$, then it must hold for all $r < r_0$, so by Lemma 4.11 and unique continuation, u must be homogeneous of degree $C(a)$. \square

The next result establishes a Weiss-type monotonicity formula, which will be an important tool in our study. Since the proof is rather technical, for readers' convenience we have moved it to the appendix.

Theorem 4.13 (Weiss-type monotonicity formula). *Let u be a minimizer of (4.1). Then the functional*

$$r \mapsto W(r) = W(r, u) := \frac{1}{r^{n-a}} \int_{B_r} |\nabla u|^2 |x_n|^a + \frac{4}{r^{n-a}} \int_{B'_r} (\lambda_+ u^+ + \lambda_- u^-) - \frac{1-a}{r^{n+1-a}} \int_{\partial B_r} u^2 |x_n|^a$$

is nondecreasing for $0 < r < 1$. Moreover, W is constant for $r \in (r_1, r_2)$ if and only if u is homogeneous of degree $1-a$ in the ring $r_1 < |x| < r_2$.

As we will see, the case $a = 0$ will require a special treatment. The following result, known as the Alt-Caffarelli-Friedman (ACF) monotonicity formula, will be instrumental in the study of that case.

Lemma 4.14 (ACF monotonicity formula). *Let $\{w_+, w_-\}$ be a pair of nonnegative continuous subharmonic functions in B_R such that $w_+ \cdot w_- = 0$ in B_R . Then the functional*

$$r \mapsto \Phi(r, w_+, w_-) := \frac{1}{r^4} \int_{B_r} \frac{|\nabla w_+|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla w_-|^2}{|x|^{n-2}}$$

is finite and nondecreasing for $0 < r < R$.

For the proof of Lemma 4.14 we refer to [CS05, PSU12], or the original paper [ACF84]. We also note that we can change the center of the ball to x_0 by replacing Φ with Φ^{x_0} given by

$$\Phi^{x_0}(r, w_+, w_-) := \frac{1}{r^4} \int_{B_r(x_0)} \frac{|\nabla w_+|^2}{|x - x_0|^{n-2}} \int_{B_r(x_0)} \frac{|\nabla w_-|^2}{|x - x_0|^{n-2}}$$

Remark 4.15. For $w_r(x) := w(rx)/r$, the functional $\Phi(r)$ enjoys the following rescaling property:

$$\Phi(r, w_+, w_-) = \Phi(1, (w_+)_r, (w_-)_r)$$

Remark 4.16. If u is a minimizer of (4.1) and $a = 0$, then $w_{\pm} = u^{\pm}$ are subharmonic so that Lemma 4.14 applies to the pair.

5. REGULARITY OF MINIMIZERS

In this section we prove the Hölder continuity of the minimizers of (4.1) for all $a \in (-1, 1)$ and $C^{1,\alpha}$ regularity along the thin space when $a \in (-1, 0)$. As we will see these regularity results are not optimal, but necessary for technical reasons. The optimal regularity is established in the next section.

Theorem 5.1 (Hölder continuity). *Let u be a minimizer of (4.1). Then $u \in C^{0,s}(B_1)$, $s = (1-a)/2$. Moreover, there exists $C = C(a, n, \lambda_{\pm}) > 0$ such that*

$$\|u\|_{C^{0,s}(B_{1/2})} \leq C \|u\|_{L^2(B_1, |x_n|^a)}.$$

Proof. The proof is almost identical with the proof of Theorem 3.1 in [All12], we spell out the details below.

By Corollary 4.7, we know that u is bounded in $B_{3/4}$. Then put

$$C_0 = 4 \max\{\lambda_+, \lambda_-\} \sup_{B_{3/4}} |u|.$$

For $r \in (0, 3/4)$, let v be the function that is a -harmonic in B_r and equals u on ∂B_r (in other words, v is the a -harmonic replacement of u on B_r). Then since u minimizes J_a on all balls inside B_1 , there holds

$$(5.1) \quad \int_{B_r} (|\nabla u|^2 - |\nabla v|^2) |x_n|^a \leq 4 \int_{B'_r} (\lambda_+ v^+ + \lambda_- v^-) \leq C_0 r^{n-1},$$

due to the estimate

$$\sup_{B_r} |v| \leq \sup_{B_r} |u|.$$

Since $u = v$ on ∂B_r , we also have

$$\int_{B_r} \nabla v \cdot (\nabla u - \nabla v) |x_n|^a = 0,$$

which implies

$$\int_{B_r} (|\nabla u|^2 - |\nabla v|^2) |x_n|^a = \int_{B_r} |\nabla u - \nabla v|^2 |x_n|^a,$$

so that (5.1) becomes

$$\int_{B_r} |\nabla u - \nabla v|^2 |x_n|^a \leq C_0 r^{n-1}.$$

This can now be iterated in a standard way (cf. Theorem 3.1 in [All12]), to imply

$$\int_{B_r} |\nabla u|^2 |x_n|^a \leq C r^{n-1},$$

for $0 < r < 3/4$, where C might also depend on the $W^{1,2}(B_{3/4}, |x_n|^a)$ -norm of u , which by the energy inequality (see Corollary 4.6) can be estimated in terms of $L^2(B_1, |x_n|^a)$ -norm of u . By Hölder's inequality this gives

$$\int_{B_r} |\nabla u| dx \leq \| |\nabla u| x_n^{a/2} \|_{L^2(B_r)} \| x_n^{-a/2} \|_{L^2(B_r)} \leq C r^{n-(1+a)/2}.$$

The estimate above is valid for all balls $B_r(x', 0)$ centered at the thin space, as long as they are contained in $B_{3/4}$. From arguments used in the proof of Morrey's embedding this yields the estimates

$$\left| u(x', 0) - \fint_{B_r(x', 0)} u \right| \leq C r^s, \quad |u(x', 0) - u(y', 0)| \leq C r^s,$$

where $s = (1 - a)/2$. Then, arguing as at the end of the proof of Theorem 3.1 in [All12] we can conclude that u is s -Hölder continuous in $B_{1/2}$ with the required estimate on the $C^{0,s}$ norm. \square

Remark 5.2. The fact that u is a minimizer of J_a in B_1 can be changed with the following conditions:

- (1) $L_a u = 0$ in B_1^+ ;

(2) $|\lim_{x_n \rightarrow 0+} x_n^a u_{x_n}(x', x_n)| \leq \mu$ for $x' \in B'_1$ in the sense that

$$\left| \int_{B_1^+} (\nabla u \cdot \nabla \psi) x_n^a \right| \leq \mu \int_{B'_1} |\psi|,$$

for any $\psi \in C_0^\infty(B_1)$ (cf. Lemma 4.1);

(3) $|u| \leq M$ in B_1^+ .

Then we will have that

$$\|u\|_{C^{0,s}(B_{1/2})} \leq C(M, \mu, n, a).$$

Instead of (5.1) one will have to use

$$\begin{aligned} \left| \int_{B_r} \nabla u (\nabla v - \nabla u) |x_n|^a \right| &\leq 2\mu \left| \int_{B'_1} |u - v| \right| \\ &\leq C(M, \mu, n) r^{n-1} \end{aligned}$$

together with

$$\int_{B_r} \nabla v (\nabla v - \nabla u) |x_n|^a = 0.$$

We now turn to the $C^{1,\alpha}$ regularity along the thin space when $a \in (0, 1)$.

Theorem 5.3 ($C^{1,\alpha}$ regularity, $a < 0$). *Let u be minimizer of (4.1) in B_1 . If $a \in (-1, 0)$, then $u \in C^{1,\alpha}(B'_1)$ for $\alpha < -a$. Moreover,*

$$\|u\|_{C^{1,\alpha}(B'_{1/2})} \leq C(M, n, a, \alpha) (\|u\|_{L^2(B_1, |x_n|^a)} + \max\{\lambda_+, \lambda_-\})$$

Proof. Let f be defined as in Lemma 4.1. We extend f to all of \mathbb{R}^{n-1} by defining $f(y') = 0$ if $y' \notin B_1$. Then $f \in L^\infty(\mathbb{R}^{n-1})$ with compact support, so we can use the Riesz potential

$$I_{2s}f(x') = c_{n,s} \int_{\mathbb{R}^{n-1}} \frac{f(y') dy'}{|x' - y'|^{n-2s}}$$

as the inverse to the fractional Laplacian in the sense that $(-\Delta_{x'})^s I_{2s}f = f$. Since $a < 0$, by Proposition 2.9 in [Sil07], $I_{2s}f \in C^{1,\alpha}(\mathbb{R}^{n-1})$ for any $\alpha < -a$. We now use the extension operator as in [CS07] on $I_{2s}f$ to obtain an a -harmonic function v in the upper half plane with Dirichlet data $I_{2s}f$, so

$$v|_{\mathbb{R}^{n-1}} = I_{2s}f.$$

We may evenly reflect v across the thin space. If $w = u - v$, then for all $\psi \in C_0^2(B_1)$

$$\int_{B'_1} |x_n|^a \nabla w \nabla \psi = 0.$$

Then $L_a w = 0$ in all of B_1 . In particular, w is C^∞ regular on the thin ball B'_1 in any x' direction (cf. Lemma 4.9). Since $u = v + w$, it follows that

$$u \in C^{1,\alpha}(B'_1) \quad \text{for } \alpha < -a.$$

The interior estimate on u follows by the interior estimate on v (see Proposition 2.9 in [Sil07]) and for the a -harmonic function w (see Lemma 4.9). \square

Remark 5.4. Similarly to Remark 5.2, in Theorem 5.3 we may change the requirement that u is a minimizer of J_a in B_1 with conditions (1)–(3) in Remark 5.2. Then, the conclusion will be that

$$\|u\|_{C^{1,\alpha}(B'_{1/2})} \leq C(M, \mu, a, \alpha, n)$$

for any $0 < \alpha < -a$.

6. OPTIMAL GROWTH AND REGULARITY

Now we have all the tools needed to prove the optimal growth of order $1 - a$, which follows below.

Theorem 6.1 (Optimal growth). *Let u be a bounded minimizer of (4.1), with $|u| \leq M$ in B_1 . Then the following holds.*

(i) *If $a \geq 0$, then*

$$\sup_{B_r(x', 0)} |u(y) - u(x', 0)| \leq Cr^{1-a}$$

for $0 < r < 1/2$ and $x' \in B'_{1/2}$, with $C = C(a, M, \lambda_{\pm}, n)$.

(ii) *If $a < 0$, then*

$$\sup_{B_r(x', 0)} |u(y) - u(x', 0) - (y - (x', 0)) \cdot \nabla_{x'} u(x', 0)| \leq Cr^{1-a},$$

for $0 < r < 1/2$ and $x' \in B'_{1/2}$, with $C = C(a, M, \lambda_{\pm}, n)$.

Proof. Case i: $a \geq 0$. For $0 < r < 1/2$ and $x' \in B'_{1/2}$ let

$$S_r(u, x') = \sup_{B_r} |u - u(x', 0)|.$$

If the assertion does not hold then we can find a sequence of minimizers u_j , points $x'_j \in B'_{1/2}$ and radii $r_j \rightarrow 0$ such that

$$\frac{S_{r_j}(u_j, x'_j)}{r_j^{1-a}} = C_j \rightarrow \infty,$$

and

$$S_{r_j 2^k}(u_j, x'_j) \leq 2^{k(1-a)} S_{r_j}(u_j, x'_j),$$

for all nonnegative integers k such that $2^k r_j \leq 1$. Let now

$$v_j(x) = \frac{u_j(r_j x + (x'_j, 0)) - u_j(x'_j, 0)}{C_j r_j^{1-a}}.$$

Then

- (1) v_j is even in x_n -variable
- (2) v_j satisfies $L_a v_j = 0$ in B_R^+ for $R < 1/r_j$;
- (3) For any $\psi \in C_0^\infty(B_R)$

$$\int_{B_R^+} (\nabla v_j \cdot \nabla \psi) x_n^a \leq \frac{\max\{\lambda_+, \lambda_-\}}{C_j} \int_{B_R'} |\psi|;$$

- (4) $\sup_{B_{2^k}} |v_j| \leq 2^{k(1-a)}$, whenever $2^k r_j \leq 1$;
- (5) $v_j(0) = 0$;
- (6) $\sup_{B_1} |v_j| = 1$.

By Theorem 5.1 and Remark 5.2, v_j is uniformly bounded in $C^{0,s}(B_R)$ for all $R < 1/r_j$. Passing to the limit, we obtain that v_j converges (over a subsequence) to a function $v_0 \in C^{0,s}(\mathbb{R}^n)$ which satisfies

$$\int_{\mathbb{R}^n} (\nabla v_0 \cdot \nabla \psi) |x_n|^a = 0$$

for any $\psi \in C_0^\infty(\mathbb{R}^n)$. In other words, v_0 is a -harmonic in \mathbb{R}^n , even in x_n -variable. Besides, we will have that

$$|v_0(x)| \leq C(1 + |x|^{1-a}), \quad v_0(0) = 0, \quad \sup_{B_1} |v_0| = 1.$$

The Liouville-type theorem in Lemma 4.10 now implies that v_0 is a polynomial.

At this point we need to split the proof into two different cases depending on a .

Subcase i.a: $a > 0$. Since the growth at infinity is less than linear v_0 must be constant and thus $v_0 \equiv 0$, which is clearly a contradiction with the properties of v_0 above.

Subcase i.b: $a = 0$. Again, due to the growth condition, we can deduce that v_0 is linear. To get a contradiction, we intend to use the ACF monotonicity formula, but in order to do so, we need to make the following observation. For the sequences u_j , x'_j and r_j in our contradictory assumption, without loss of generality we may assume that the following holds:

- (α) either $u_j(x'_j, 0) = 0$ for all j ,
- (β) $B'_{r_j}(x'_j) \subset \{u_j > 0\}$ for all j , or
- (γ) $B'_{r_j}(x'_j) \subset \{u_j < 0\}$ for all j .

Indeed, in the case when $x'_j \notin \Gamma_{u_j}$ and $r_j < \text{dist}(x'_j, \Gamma_{u_j})$, we may simply replace x'_j with its closest point $y'_j \in \Gamma_{u_j}$ and r_j with $2r_j$ and convert it to case (α).

Cases (α)–(γ) translate into the following ones for functions v_j .

- (α') $v_j(0, 0) = 0$ for all j ,
- (β') $v_j > 0$ in B'_1 for all j ,
- (γ') $v_j < 0$ in B'_1 for all j .

In the cases (β') and (γ') we obtain immediately that the limit v_0 has a sign in B'_1 . Since we also know that v_0 is linear and even in x_n , this immediately implies that v_0 must be identically zero, a contradiction with property (6) of v_j . In the case (α'), which is the same as (α), we observe that we have the additional property that

$$\Delta(u_j^\pm) \geq 0 \quad \text{in } B_1,$$

see Lemma 4.5. Then we can apply the ACF monotonicity formula (Lemma 4.14). We will have that for any $s > 0$ and large j

$$\begin{aligned} \Phi(s, v_j^+, v_j^-) &= \frac{1}{C_j^2} \Phi^{x_j}(sr_j, u_j^+, u_j^-) \\ &\leq \frac{1}{C_j^2} \Phi^{x_j}(1/4, u_j^+, u_j^-) \leq \frac{C(n)}{C_j^2} \|u_j\|_{L^2(B_1)}^2 \rightarrow 0. \end{aligned}$$

Hence, $\Phi(s, v_0^+, v_0^-) = 0$ for $s > 0$, and we can conclude that either $v_0^+ = 0$ or $v_0^- = 0$. Since v_0 is linear and even in x_n , we again conclude that v_0 is identically zero, which contradicts property (6) of v_j .

Case ii: $a < 0$. Here we define instead

$$S_r(u, x') = \sup_{y \in B_r(x', 0)} |u(y) - u(x', 0) - y' \cdot \nabla_{x'} u(x', 0)|.$$

If the assertion does not hold then we can find a sequence of minimizers u_j , points $x'_j \in B'_{1/2}$ and a radii $r_j \rightarrow 0$ such that

$$\frac{S_{r_j}(u_j, x'_j)}{r_j^{1-a}} = C_j \rightarrow \infty,$$

and

$$S_{r_j 2^k}(u_j, x'_j) \leq 2^{k(1-a)} S_{r_j}(u_j, x'_j),$$

for all nonnegative integers k such that $2^k r_j \leq 1$. Let now

$$v_j(x) = \frac{u_j(r_j x + (x', 0)) - u_j(x', 0) - (r_j x') \cdot \nabla_{x'} u_j(x'_j, 0)}{C_j r_j^{1-a}}.$$

Then, again as before,

- (1) v_j is even in x_n -variable
- (2) v_j satisfies $L_a v_j = 0$ in B_R^+ for $R < 1/r_j$;
- (3) For any $\psi \in C_0^\infty(B_R)$

$$\int_{B_R^+} (\nabla v_j \cdot \nabla \psi) x_n^a \leq \frac{\max\{\lambda_+, \lambda_-\}}{C_j} \int_{B_R'} |\psi|;$$

- (4) $\sup_{B_{2^k}} |v_j| \leq 2^{k(1-a)}$, whenever $2^k r_j \leq 1$;
- (5) $v_j(0) = |\nabla_{x'} v_j(0)| = 0$;
- (6) $\sup_{B_1} |v_j| = 1$.

By Theorem 5.1, Remark 5.2, Theorem 5.3 and Remark 5.4, there is a subsequence, again labelled v_j , converging in $C_{\text{loc}}^{0,s} \cap C_{\text{loc}}^{1,\alpha}(\mathbb{R}^{n-1} \times 0)$ and then by (3) above also weakly in $W_{\text{loc}}^{1,2}(|x_n|^a)$ to a limit function v_0 . By passing to the limit in (3) and (4) above, we see that

$$L_a v_0 = 0 \quad \text{in } \mathbb{R}^n$$

and

$$|v_0(x)| \leq C(1 + |x|^{1-a}), \quad v_0(0) = |\nabla_{x'} v_0(0)| = 0, \quad \sup_{B_1} |v_0| = 1.$$

Since v_0 is also even in x_n variable, Lemma 4.10 now implies that v_0 is a polynomial. Since the growth at infinity is less than quadratic, v_0 must be linear. The properties of v_0 above imply that $v_0 \equiv 0$, a contradiction. \square

With the proposition above, we can now prove the optimal regularity. More precisely, we prove the following version of Theorem 1.2.

Theorem 6.2 (Optimal regularity). *Let u be a bounded minimizer of (4.1), with $0 \in \Gamma^+ \cup \Gamma^-$ and $|u| \leq M$ in B_1 . Then the following holds.*

- (i) *If $a \geq 0$, then $u \in C^{0,1-a}(B_{1/4})$ with*

$$\|u\|_{C^{0,1-a}(B_{1/4})} \leq C(a, M, \lambda_\pm, n).$$

- (ii) *If $a < 0$, then $u \in C^{1,-a}(B_{1/4}^\pm \cup B'_{1/4})$ with*

$$\|u\|_{C^{1,-a}(B_{1/4}^\pm \cup B'_{1/4})} \leq C(a, M, \lambda_\pm, n).$$

Proof. Case i: $a \geq 0$. First, from the growth estimate in Theorem 6.1 and Lemma 4.9 applied in the ball $B_{x_n}(x', x_n)$, we obtain that

$$(6.1) \quad |\nabla_{x'} u(x)| \leq C x_n^{-a}, \quad \text{for } x \in B_{1/4}^+,$$

where $C = C(a, M, \lambda_\pm, n)$. Besides, we claim that a similar estimate holds also in x_n -direction:

$$(6.2) \quad |u_{x_n}(x)| \leq C x_n^{-a}, \quad \text{for } x \in B_{1/4}^+.$$

To show this estimate, we note that from Lemma 4.1, we know that the function $\tilde{u} = x_n^a u_{x_n}$ is uniformly bounded in $B'_{1/2}$, with the bound depending on λ^\pm . Moreover, it is known that $L_{-a}\tilde{u} = 0$ in B_1^+ (see for instance [CS07] for an explanation). Next, let v be the bounded solution of

$$L_{-a}v = 0 \quad \text{in } \mathbb{R}_+^n, \quad v(x', 0) = \tilde{u}(x', 0)\zeta(x'), \quad x' \in \mathbb{R}^{n-1},$$

with some cutoff function $\zeta \in C_0^\infty(B'_1)$ which equals 1 on $B'_{1/2}$. Then, by using the Poisson formula (see [CS07]), we will have

$$\|v\|_{L^\infty(B_{1/2}^+)} \leq C \|\tilde{u}\zeta\|_{L^\infty(B'_{1/2})} \leq C(a, M, \lambda^\pm, n).$$

Furthermore, let $w = u - v$ and extend w in an odd manner across the thin space. Then, w is $(-a)$ -harmonic in $B_{1/2}$. Hence, from Lemma 4.5 and Corollaries 4.6 and 4.7, we have

$$\begin{aligned} \|w\|_{L^\infty(B_{1/2}^+)} &\leq C \|w\|_{L^2(B_{3/4}, |x_n|^{-a})} \\ &= C \|\nabla u\|_{L^2(B_{3/4}, |x_n|^a)} \leq C \sup_{B_1} |u| = CM. \end{aligned}$$

Combining the estimates for v and w , we obtain a similar estimate for \tilde{u} , which implies (6.2).

With estimates (6.1)–(6.2), we can now prove the Hölder continuity of u . Let $x, y \in B_{1/4}^+ \cup B'_{1/4}$ and without loss of generality assume that $x_n \leq y_n$. Consider the following possibilities:

(1) $|x - y| \geq y_n/2$. In this case, by Theorem 6.1, we have

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x', 0) - u(y', 0)| + |u(x) - u(x', 0)| + |u(y) - u(y', 0)| \\ &\leq C|x' - y'|^{1-a} + Cx_n^{1-a} + Cy_n^{1-a} \leq C|x - y|^{1-a}. \end{aligned}$$

(2) $|x - y| < y_n/2$. In this case, $x, y \in B_{y_n/2}(y)$ and we have that by (6.1)–(6.2)

$$|\nabla u| \leq C y_n^{-a} \quad \text{in } B_{y_n/2}(y).$$

Hence

$$|u(x) - u(y)| \leq C y_n^{-a} |x - y| \leq C |x - y|^{1-a}.$$

Thus, we obtain the desired Hölder continuity of u in $B_{1/4}^+ \cup B'_{1/4}$. By symmetry in x_n variable, we therefore have the same Hölder continuity in $B_{1/4}$.

Case ii: $a < 0$. First, note that the estimate (6.2) still holds in this case. To prove the desired $C^{1,-a}$ regularity we will similar estimates for the second derivatives.

To this end, take a point $x = (x', x_n) \in B_{1/4}^+$. Then $\tilde{u} = x_n^a u_{x_n}$ is $(-a)$ -harmonic in $B_{x_n}(x)$ and applying Lemma 4.9 we obtain

$$(6.3) \quad |\nabla_{x'} u_{x_n}(x)| \leq C x_n^{-1-a} \quad \text{for } x \in B_{1/4}^+.$$

Further, using that $w(y) = u(y) - u(x', 0) - (y - (x', 0)) \cdot \nabla_{x'} u(x', 0)$, is a -harmonic in $B_{x_n}(x)$, we obtain by Lemma 4.9

$$(6.4) \quad |D_{x'}^2 u(x)| = |D_{x'}^2 w(x)| \leq \frac{C}{x_n^2} \sup_{B_{x_n/2}(x)} |w| \leq C x_n^{-1-a}, \quad \text{for } x \in B_{1/4}^+,$$

where the last inequality follows from the optimal growth estimate in Theorem 6.1. Now, using the equation for u , we conclude that

$$(6.5) \quad |u_{x_n x_n}(x)| = |-\Delta_{x'} u(x) - a x_n^{-1} u_{x_n}(x)| \leq C x_n^{-1-a}, \quad \text{for } x \in B_{1/4}^+.$$

Combining (6.3)–(6.5), we arrive at

$$(6.6) \quad |D^2 u(x)| \leq C x_n^{-1-a}, \quad \text{for } x \in B_{1/4}^+.$$

Now taking $x, y \in B_{1/4}^+ \cup B'_{1/4}$ and repeating the arguments as at the end of Case i, we readily conclude that

$$|\nabla u(x) - \nabla u(y)| \leq C |x - y|^{-a}.$$

(In the estimate of u_{x_n} we use that u_{x_n} is identically zero on $B'_{1/2}$, by estimate (6.2).) Furthermore, using Theorem 5.3, integrating (6.3) from $(x', 0)$ to (x', x_n) , and using (6.2) one more time, we also establish that

$$|\nabla u| \leq C \quad \text{for } x \in B_{1/4}^+.$$

Hence,

$$\|u\|_{C^{1,-a}(B_{1/4}^+ \cup B'_{1/4})} \leq C(a, M, \lambda^\pm, n),$$

as required. \square

The proof of Theorem 1.2 is now immediate.

Proof of Theorem 1.2. . The proof follows from the local boundedness (see Corollary 4.7), Theorem 6.2, and a simple covering argument by balls. \square

7. NONDEGENERACY

In this section we prove a certain nondegeneracy property for the minimizers that will be instrumental for the proof of our second main result, Theorem 1.3. We follow the outline as given in [AP12].

Theorem 7.1 (Nondegeneracy). *Fix $0 < t < 1$, and let u be a minimizer of (4.1). There exists $\epsilon > 0$ with ϵ depending only on λ_\pm and t , such that if $u|_{\partial B_r} \leq \epsilon r^{1-a}$ ($u|_{\partial B_r} \geq -\epsilon r^{1-a}$) then*

$$u(x) \leq 0 \quad (u(x) \geq 0) \quad \text{for } x \in B'_{tr}.$$

We will need the following lemma, which is along the lines of Corollary 4.4.

Lemma 7.2. *Let u be a minimizer of (4.1) such that $u|_{\partial B_1} = M$. Then u is symmetric about the line $(0, \dots, 0, x_n)$ and*

$$u(x', 0) = f(|x'|)$$

where f is a nondecreasing function. Therefore, if the coincidence set is nonempty then the coincidence set $\{u = 0\} = \overline{B'_\rho}$ for some $0 \leq \rho \leq 1$.

Proof. Extend u to be a function on the cube Q with side length 2, by defining $u(x) = M$ for $x \notin B_1$. We now apply Steiner symmetrization (as defined in [Kaw85, page 82]) to the function $w = M - u$ on lines parallel to $\mathbb{R}^{n-1} \times \{0\}$. If we only consider $\{x \mid |x_n| > \epsilon\}$, then w is Lipschitz. Then by [Kaw85, page 82], if we Steiner symmetrize w to obtain v deduce for each $\epsilon > 0$:

$$\int_{B_1 \cap \{|x_n| > \epsilon\}} |\nabla u|^2 |x_n|^a = \int_{B_1 \cap \{|x_n| > \epsilon\}} |\nabla w|^2 |x_n|^a \geq \int_{B_1 \cap \{|x_n| > \epsilon\}} |\nabla v|^2 |x_n|^a$$

v will have the same boundary values as w on ∂B_1 . Then by letting $\epsilon \rightarrow 0$ we obtain

$$\int_{B_1} |\nabla u|^2 |x_n|^a = \int_{B_1} |\nabla w|^2 |x_n|^a \geq \int_{B_1} |\nabla v|^2 |x_n|^a.$$

Finally, we note that

$$\int_{B'_1} \lambda_+ u$$

is invariant under Steiner symmetrization. Since minimizers are unique, we see that our minimizer is Steiner symmetric about the line $(0, \dots, 0, x_n)$ and $\{u = 0\}$ is connected and centered at the origin. \square

Lemma 7.3. *Let u_ϵ be the minimizer of (4.1) subject to the boundary conditions $u_\epsilon \equiv \epsilon$ on ∂B_1 . Then $u_\epsilon \equiv 0$ on B'_ρ for some $\rho = \rho(\epsilon) > 0$. Furthermore $\rho(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$.*

Proof. It is clear that the minimizer $u_\epsilon \geq 0$. From the comparison principle $u_{\epsilon_1} \leq u_{\epsilon_2}$ if $\epsilon_1 \leq \epsilon_2$. Suppose by way of contradiction that $u_\epsilon(x', 0) > 0$ for all ϵ and for x' in the ring

$$\mathcal{R} = \{(x', 0) \in B'_1 \mid 0 \leq r_1 < |x'| < r_2 \leq 1\}.$$

By Lemma 4.1

$$\lim_{t \rightarrow 0} t^a \frac{\partial u_\epsilon}{\partial x_n}(x', t) = \lambda_+.$$

Thus

$$u_\epsilon - \frac{\lambda_+}{1-a} |x_n|^{1-a}$$

will be a -harmonic in the tube

$$\mathcal{T} = \{x \in B_1 \mid 0 \leq r_1 < |x'| < r_2 \leq 1\}.$$

Now, $u_\epsilon \rightarrow u \equiv 0$ uniformly, which implies that $-\frac{\lambda_+}{1-a} |x_n|^{1-a}$ is a -harmonic in \mathcal{T} . This is clearly a contradiction. Thus, there exists ϵ and $y' \in \mathcal{R}$ such that $u_\epsilon(y', 0) = 0$. Then, by Lemma 7.2, $u_\epsilon \equiv 0$ on B'_{r_1} . \square

We are now able to prove the nondegeneracy result.

Proof of Theorem 7.1. First we note that by rescaling we only need to prove Theorem 7.1 on the unit ball B_1 . Pick $0 < t < 1$. Lemma 7.3 proves that there exists ϵ depending on t and λ_+ such that $\{u_\epsilon = 0\} = B'_t$. By the comparison principle if u is a minimizer such that $u \leq u_\epsilon = \epsilon$ on ∂B_1 , then $u \leq 0$ on B'_t . The case for which $u \geq -\epsilon$ is proven similarly. \square

Theorem 7.1 immediately implies the corollary below, which is the result we will be using later on.

Corollary 7.4. *If u is a minimizer and $0 \in \Gamma^+$ ($0 \in \Gamma^-$), then*

$$\sup_{\partial B_r} u \geq Cr^{1-a} \quad \left(\inf_{\partial B_r} u \leq -Cr^{1-a} \right),$$

where C depends only on a , λ_{\pm} and n .

8. BLOWUPS AND GLOBAL SOLUTIONS

In this section we study the blowups and global solutions that will be the main step towards proving Theorem 1.3.

Let u be a minimizer of (4.1) and assume that $0 \in \Gamma^+ \cup \Gamma^-$. For $r > 0$ consider the rescalings

$$u_r(x) = \frac{u(rx)}{r^{1-a}}, \quad x \in B_{1/r}.$$

From the scaling properties of J_a , it is easy to see that u_r is a minimizer of J_a in $B_{1/r}$. By the optimal growth estimate we will then have that

$$\sup_{B_R} |u_r| \leq CR^{1-a}, \quad (r < 1/R)$$

when $a \geq 0$, so the family $\{u_r\}_{0 < r < 1}$ is locally bounded in \mathbb{R}^n . Note that to have the same conclusion when $a < 0$ we must assume additionally that $|\nabla_{x'} u(0)| = 0$. The local boundedness implies boundedness in $W_{\text{loc}}^{1,2}(\mathbb{R}^n, |x_n|^a)$, by the energy inequality. Thus, over a sequence $r_k \rightarrow 0$ the rescalings u_{r_k} will converge to a certain u_0 weakly in $W_{\text{loc}}^{1,2}(\mathbb{R}^n, |x_n|^a)$. (This convergence is actually strong, as we prove in Lemma 8.1 below.) Passing to a subsequence we may assume that the convergence is strong in $L_{\text{loc}}^2(\mathbb{R}^n, |x_n|^a)$ and $L_{\text{loc}}^2(\mathbb{R}^{n-1} \times \{0\})$. We call such u_0 a blowup of u at the origin. A standard argument also shows that the blowup u_0 will be a minimizer of J_a on any $U \Subset \mathbb{R}^n$. Such minimizer we will call *global minimizers* of J_a .

Our analysis of blowups starts with the following lemma on the strong convergence of minimizers.

Lemma 8.1 (Strong convergence). *Let $\{u_k\}$ be a sequence of minimizers of J_a in D^+ that converges weakly in $W^{1,2}(D^+, x_n^a)$ to some minimizer u . Then, over a subsequence, u_k converges strongly to u in $W^{1,2}(U^+, x_n^a)$ for any $U \Subset D$.*

Proof. Take a test function $\eta \in C_0^\infty(D)$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in a neighborhood of } \overline{U}.$$

Then, integrating by parts we have

$$\begin{aligned} \int_{D^+} |\nabla(u_k - u)|^2 \eta^2 |x_n|^a &= \int_{D'} (f_k - f)(u_k - u) \eta^2 \\ &\quad - 2 \int_{D^+} (u_k - u) \eta \nabla \eta \nabla(u_k - u) |x_n|^a, \end{aligned}$$

where $f_k(x') = -\lim_{x_n \rightarrow 0^+} x_n^a \partial_{x_n} u_k(x', x_n)$ and $f(x') = -\lim_{x_n \rightarrow 0^+} x_n^a \partial_{x_n} u(x', x_n)$ in the sense of Lemma 4.1, which also tells that f_k and f are uniformly bounded by $\max\{\lambda_{\pm}\}$. Using that bound and applying the Young's inequality to the second integral on the right-hand side in the usual manner, we obtain that

$$\int_{U^+} |\nabla(u_k - u)|^2 |x_n|^a \leq C \int_{D' \cap K} |u_k - u| + C \int_{D^+ \cap K} (u_k - u)^2 |x_n|^a,$$

where $K = \text{supp } \eta \Subset D$. The proof now follows from the strong convergence of u_k to u in $L^2(D' \cap K)$ and $L^2(D^+ \cap K, |x_n|^a)$ by the compactness of the trace operator and the Sobolev embedding. \square

Having the strong convergence of minimizers combined with the Weiss-type monotonicity formula, we immediately obtain the following property of blowups.

Lemma 8.2. *Let u be a minimizer of (4.1) with $0 \in \Gamma^+ \cup \Gamma^-$. If $a < 0$ make the further assumption that $|\nabla_{x'} u(0)| = 0$. Suppose also that u_{r_k} converges to a blowup u_0 . Then u_0 is homogeneous of degree $(1 - a)$.*

Proof. We will use the following scaling property of the Weiss energy functional

$$W(\rho r, u) = W(\rho, u_r).$$

Now, by Lemma 8.1, we may assume that $u_{r_k} \rightarrow u_0$ strongly in $W^{1,2}(U, |x_n|^a)$ for any $U \Subset \mathbb{R}^n$ and we can pass in the limit in the Weiss energy functional to obtain

$$W(\rho, u_0) = \lim_{k \rightarrow \infty} W(\rho, u_{r_k}) = \lim_{k \rightarrow \infty} W(\rho r_k, u) = W(0+, u)$$

for any $\rho > 0$. This implies that $W(\cdot, u_0) \equiv \text{const} = W(0+, u)$ and consequently that u_0 is homogeneous of degree $(1 - a)$, by the second part of Theorem 4.13. \square

Lemma 8.3 (Homogeneous global solutions). *Let u be a global minimizer of J_a and assume u is homogeneous of degree $(1 - a)$. Then $u = c|x_n|^{1-a}$ for some constant c .*

Proof. Choose c to be such that if $v = u - c|x_n|^{1-a}$, then

$$(8.1) \quad \int_{\partial B_1} |x_n|^a v = 0.$$

We claim that

$$\frac{-\lambda_-}{(1-a)} \leq c \leq \frac{\lambda_+}{(1-a)}.$$

The heuristic idea behind the claim is that if c is too large then $L_a v \leq 0$ in all of B_1 , and so $v(0) > 0$. To proceed proving this claim we use the definition of f in as given in Lemma 4.1. Suppose that $c > \lambda_+/(1-a)$. Using that $\text{div}(|x_n|^a \nabla v) = 0$ off the thin spaces, we integrate by parts to obtain

$$(8.2) \quad \int_{\partial B_r} |x_n|^a v_\nu = 2 \int_{B'_r} (f - c(1-a)) < 0,$$

since $f \leq \lambda_+ \leq c(1-a)$. Also, for almost every r

$$\frac{d}{dr} \left[\frac{1}{r^{n-1+a}} \int_{\partial B_r} |x_n|^a v \right] = \int_{\partial B_r} |x_n|^a v_\nu.$$

It follows that

$$\frac{1}{r^{n-1+a}} \int_{\partial B_r} |x_n|^a v \quad \text{is decreasing in } r.$$

Then $v(0) > 0$. This is a contradiction since $u(0) = 0$, and consequently $v(0) = 0$. A similar argument shows that $c \geq -\lambda_-/(1-a)$.

Now we use integration by parts to obtain

$$\int_{B_r} |x_n|^a |\nabla v|^2 = \int_{\partial B_r} |x_n|^a v v_\nu - 2 \int_{B'_r} (f - c(1-a)) v.$$

By homogeneity of v we obtain that $v_\nu = (1-a)v/r$. We now turn to simplifying the integral along the thin space. Note that $v = u$ on the thin space. Also by Lemma 4.1, we may write $f(y')$ explicitly when $u(y', 0) \neq 0$. Since we are multiplying by u on the thin space we only need to consider the case when $u(y', 0) \neq 0$. So our equality above becomes

$$(8.3) \quad \int_{B_r} |x_n|^a |\nabla v|^2 = \frac{1-a}{r} \int_{\partial B_r} |x_n|^a v^2 - 2 \int_{B'_r} (\lambda_+ - c(1-a))u^+ + (\lambda_- + c(1-a))u^-$$

We then define $c_+ = (\lambda_+ - c(1-a))/2$ and $c_- = (\lambda_- + c(1-a))/2$. Now $c_\pm \geq 0$ by our claim above. Also either $c_+ > 0$ or $c_- > 0$. If we move everything to the left hand side in (8.3) and multiply by r , we obtain

$$(8.4) \quad r \left(\int_{B_r} |x_n|^a |\nabla v|^2 + 4 \int_{B'_r} (c_+ v^+ + c_- v^-) \right) - (1-a) \int_{\partial B_r} |x_n|^a v^2 = 0.$$

If w is the a -harmonic replacement of v in B_r , then $w(0) = 0$ by (8.1) and w is even in x_n since v is even. Now suppose v is not equivalently zero. From Corollary 4.12 we obtain

$$1 \leq N(w, r) \leq N(v, r).$$

Combining the above inequality with (8.4) we have

$$1 \leq N(w, r) \leq N(v, r) \leq r \frac{\int_{B_r} |x_n|^a |\nabla v|^2 + 4 \int_{B'_r} (c_+ v^+ + c_- v^-)}{\int_{\partial B_r} |x_n|^a v^2} = 1 - a.$$

If $a > 0$, then we obtain an immediate contradiction, and thus for $a > 0$, $u \equiv c|x_n|^{1-a}$. If $a = 0$, then necessarily

$$\int_{B'_r} (c_+ v^+ + c_- v^-) = 0.$$

Then $v(x', 0) \equiv 0$. By using an odd reflection in x_n variable, we obtain a harmonic function, homogeneous of degree 1 in all of \mathbb{R}^n . Thus, by the Liouville theorem we conclude that $v = c_3 x_n$ if $x_n > 0$ and by even symmetry then $v = c_3 |x_n|$ for all x . By (8.1) we must have $c_3 = 0$ and so $v \equiv 0$.

We now turn to the case in which $-1 < a < 0$. Let $w = v - \ell$ where ℓ is a linear function in the first $n-1$ directions such that the a -harmonic replacement of w in B_1 has zero gradient in the thin space. Then $w(0) = 0$ and the equality (8.1) holds for w .

By using Corollary 4.12 on the a -harmonic replacement of w , we obtain $2 \leq N(w, r)$. Then

$$\begin{aligned} 0 &\leq r \int_{B_r} |x_n|^a |\nabla w|^2 - 2 \int_{\partial B_r} |x_n|^a w^2 \\ &= r \left(\int_{B_r} |x_n|^a |\nabla v|^2 - 2 \int_{B_r} |x_n|^a \nabla v \nabla \ell + \int_{B_r} |x_n|^a |\nabla \ell|^2 \right) \\ &\quad - 2 \int_{\partial B_r} |x_n|^a v^2 + 4 \int_{\partial B_r} |x_n|^a v \ell - 2 \int_{\partial B_r} |x_n|^a \ell^2. \end{aligned}$$

Using integration by parts together with the fact that $L_a \ell = 0$ and ℓ is homogeneous of degree 1, we obtain

$$\begin{aligned} 0 &\leq r \left(\int_{B_r} |x_n|^a |\nabla v|^2 + \int_{B_r} |x_n|^a |\nabla \ell|^2 \right) \\ &\quad - 2 \int_{\partial B_r} |x_n|^a v^2 + 2 \int_{\partial B_r} |x_n|^a v \ell - 2 \int_{\partial B_r} |x_n|^a \ell^2. \end{aligned}$$

From (8.4) we conclude

$$\begin{aligned} 0 &\leq -4r \int_{B'_r} (c_+ v^+ + c_- v^-) - (1+a) \int_{\partial B_r} |x_n|^a v^2 \\ &\quad + 2 \int_{\partial B_r} |x_n|^a v \ell - \int_{\partial B_r} |x_n|^a \ell^2 \\ &\quad + r \int_{B_r} |x_n|^a |\nabla \ell|^2 - \int_{\partial B_r} |x_n|^a \ell^2 \\ &= -4r \int_{B'_r} (c_+ v^+ + c_- v^-) - (1+a) \int_{B_r} |x_n|^a v^2 \\ &\quad + 2 \int_{\partial B_r} |x_n|^a v \ell - \int_{\partial B_r} |x_n|^a \ell^2 + 1. \end{aligned}$$

Using the homogeneity of v and ℓ , scaling and multiplying by $r^{-(n+1-a)}$, the above inequality becomes

$$\begin{aligned} 0 &\leq -4 \int_{B'_1} (c_+ v^+ + c_- v^-) - (1+a) \int_{\partial B_1} |x_n|^a v^2 \\ &\quad + 2r^a \int_{\partial B_1} |x_n|^a v \ell - r^{2a} \int_{\partial B_1} |x_n|^a \ell^2 + r^{-(n+1-a)}. \end{aligned}$$

This inequality holds for every $r > 0$. Passing $r \rightarrow \infty$ yields

$$0 \leq -4 \int_{B'_1} (c_+ v^+ + c_- v^-) - (1+a) \int_{\partial B_1} |x_n|^a v^2.$$

This clearly implies that $v \equiv 0$, which is equivalent to the statement of the lemma. \square

We conclude this section with the following generalization of Lemma 8.3, which will be important in the next section when studying the limits of rescalings of possibly different functions, when we don't have the homogeneity, but can control the growth at infinity.

Lemma 8.4 (Global minimizers). *Let u be a global minimizer of J_a such that $0 \in \Gamma^+ \cup \Gamma^-$ and $|\nabla_{x'} u(0)| = 0$ if $a < 0$. Assume also that*

$$|u(x)| \leq C(1 + |x|^{1-a}).$$

Then $u = c|x_n|^{1-a}$ for some constant c .

Proof. We first perform a blow-up $u_{r_j} \rightarrow u_0$, and conclude by Lemma 8.3 that $u_0 = c_0|x_n|^{1-a}$. Next, considering again the usual rescalings

$$u_R(x) = \frac{u(Rx)}{R^{1-a}}$$

and noticing that u_R are also locally uniformly bounded as $R \rightarrow \infty$, we can extract a subsequence $R_j \rightarrow \infty$ such that $u_{R_j} \rightarrow u_\infty$. Arguing precisely as in Lemma 8.2, we obtain that u_∞ is homogeneous and thus, by Lemma 8.3, $u_\infty = c_\infty |x_n|^{1-a}$.

On the other hand, we can compute that $W(r, |x_n|^{1-a}) \equiv 0$, which implies

$$0 = W(1, u_0) = \lim_{r \rightarrow 0} W(r, u) \leq W(s, u) \leq \lim_{R \rightarrow \infty} W(R, u) = W(1, u_\infty) = 0,$$

for any $s > 0$. Hence, $W(u, s) \equiv 0$ and u is homogeneous, and again by Lemma 8.3, $u = c|x_n|^{1-a}$. \square

9. STRUCTURE OF THE FREE BOUNDARY

In this section we prove Theorem 1.3 in the following more expanded form.

Theorem 9.1 (Structure of the free boundary). *Let u be a bounded minimizer of (4.1) such that $|u| \leq M$ in B_1 .*

(i) *If $a \geq 0$, then*

$$\Gamma^+ \cap \Gamma^- = \emptyset.$$

More specifically, there is $c_0 = c(a, M, \lambda_\pm, n) > 0$ such that if $x_0 \in \Gamma^+ \cap B'_{1/2}$ then $B'_{c_0}(x_0) \cap \Gamma^- = \emptyset$.

(ii) *If $a < 0$, then*

$$|\nabla_{x'} u| \neq 0 \quad \text{on } \Gamma^+ \cap \Gamma^-.$$

More specifically, there is $c_0 = c(a, M, \lambda_\pm, n)$ such that if $x_0 \in \Gamma^+ \cap B_{1/2}$ and $|\nabla_{x'} u(x_0)| = 0$, then $B_{c_0}(x_0) \cap \Gamma^- = \emptyset$.

Proof. Let $x_0 \in \Gamma_u^+ \cap B_{1/2}$ (with an additional property that $|\nabla_{x'} u(x_0)| = 0$ if $a < 0$). By translation and rescaling, we may assume that $x_0 = 0$. Then, if the statement of the theorem fails, then there exists a sequence of minimizers u_j of (4.1) satisfying

$$|u_j| \leq M, \quad 0 \in \Gamma_{u_j}^+, \quad |\nabla_{x'} u_j(0)| = 0 \quad \text{if } a < 0.$$

together with a sequence of points $x^j \in \Gamma_{u_j}^-$ such that $r_j = |x^j| \rightarrow 0$. We will show that this is impossible. Let

$$v_j(x) = (u_j)_{r_j}(x) = \frac{u_j(r_j x)}{r_j^{1-a}}.$$

By Theorem 5.1 combined with Theorem 6.1, $\{v_j\}$ is uniformly bounded in $C^\alpha(B_R) \cap W^{1,2}(B_R, |x_n|^a)$ for every ball B_R . Moreover, we know:

(1) v_j is a minimizer of J_a in $B_R \subset B_{1/r_j}$.

(2) By Theorem 6.1 we have the growth estimates

$$|v_j(x)| \leq C|x|^{1-a}, \quad |x| \leq 1/(2r_j).$$

(3) Since $0 \in \Gamma_{v_j}^+$ and $z^j = x^j/r_j \in \Gamma_{v_j}^- \cap B'_1$, by Theorem 7.1 we have

$$\sup_{B_t} v_j \geq Ct^{1-a}, \quad \inf_{B_t(z^j)} v_j \leq -Ct^{1-a}.$$

Consequently, we can extract a subsequence, again labeled v_j , converging locally uniformly to v_0 , a global minimizer of J_a . Due to the nondegeneracy property (3) above, we also know that v_0 has nontrivial positive and negative phases. On the other hand, by (2) we also know that

$$|v_0(x)| \leq C|x|^{1-a}, \quad \text{for any } x \in \mathbb{R}^n$$

and thus from Lemma 8.4 we conclude that $v_0 = c|x_n|^{1-a}$. Clearly, this is not possible. \square

Theorem 1.3 now follows.

Proof of Theorem 1.3. This is an immediate consequence of the local boundedness (see Corollary 4.7), Theorem 9.1, and a simple covering argument by balls. \square

We conclude this section and the paper by providing an example showing that when $a < 0$ it is possible for the two phases to meet. This is done by constructing a minimizer whose Weiss energy at a point is negative.

Example 9.2 (The two phases can meet when $a < 0$). For simplicity in our construction we will assume $\lambda_+ = \lambda_- = \lambda$. Fix some $i \in \{1, \dots, n-1\}$ and let $\ell(x) = Mx_i$. Then consider the minimizer of the functional J_a with boundary values given by $u = \ell$ on ∂B_1 . By symmetry of the boundary values of u on ∂B_1 combined with our choice of $\lambda_+ = \lambda_-$ and the fact that minimizers are unique, it is clear that $u(0) = 0$. Now we look at the Weiss energy of ℓ in B_1 . By homogeneity of ℓ

$$W(1, \ell) = M \int_{B'_1} \lambda |x_i| + aM^2 \int_{\partial B_1} x_i^2 |x_n|^a,$$

and we see that for large enough M (since $a < 0$), $W(1, \ell) < 0$. Since u and ℓ coincide on ∂B_1 , this implies that $W(1, u) < 0$. Now we argue that $0 \in \Gamma_u^+ \cup \Gamma_u^-$. Indeed, otherwise then $u \equiv 0$ in B'_ρ for some small ρ (recall $u(0) = 0$). If we reflect u by odd reflection across the thin space we obtain \tilde{u} which is a -harmonic in the solid ball B_ρ so that $W(\rho, u) = W(\rho, \tilde{u})$. By Corollary 4.12, $W(\rho, \tilde{u}) \geq 0$, contradicting $W(\rho, u) \leq W(1, u) < 0$. Thus, $0 \in \Gamma_u^+ \cup \Gamma_u^-$. We further claim that $|\nabla_{x'} u(0)| > 0$. Indeed, if $|\nabla_{x'} u(0)| = 0$, by taking a blowup at 0, we will have that $W(0+, u) = W(1, u_0) = 0$ (see Lemma 8.3). This is clearly a contradiction. Thus, $|\nabla_{x'} u(0)| > 0$, and consequently $0 \in \Gamma_u^+ \cap \Gamma_u^-$.

APPENDIX A. PROOF OF WEISS MONOTONICITY FORMULA

Proof of Theorem 4.13. The proof is very similar to that of Theorem 4.3 in [All12] and we omit the details that are derived identically.

Using that u is a minimizer of (4.1), we obtain that for a.e. $r \in (0, 1)$ we have the equality

$$\begin{aligned} 0 &= (n-2+a) \int_{B_r} |x_n|^a |\nabla u|^2 - r \int_{\partial B_r} |x_n|^a (|\nabla u|^2 - 2u_\nu^2) \\ &\quad + 4(n-1) \int_{B'_r} (\lambda_+ u^+ + \lambda_- u^-) - 4r \int_{\partial B'_r} (\lambda_+ u^+ + \lambda_- u^-) \\ (A.1) \quad &= (n-1) \int_{B_r} |x_n|^a |\nabla u|^2 - r \int_{\partial B_r} |x_n|^a |\nabla u|^2 \\ &\quad + 4(n-1) \int_{B'_r} (\lambda_+ u^+ + \lambda_- u^-) - 4r \int_{\partial B'_r} (\lambda_+ u^+ + \lambda_- u^-) \\ &\quad - (1-a) \int_{B_r} |x_n|^a |\nabla u|^2 + 2r \int_{\partial B_r} |x_n|^a u_\nu^2. \end{aligned}$$

By Lemma 4.1,

$$\int_{B_r} |x_n|^a |\nabla u|^2 = \int_{\partial B_r} |x_n|^a u u_\nu + 2 \int_{B'_r} (\lambda_+ u^+ + \lambda_- u^-),$$

and hence

$$\begin{aligned} 0 &= (n-a) \int_{B_r} |x_n|^a |\nabla u|^2 - r \int_{\partial B_r} |x_n|^a |\nabla u|^2 \\ &\quad + 4(n-a) \int_{B'_r} (\lambda_+ u^+ + \lambda_- u^-) - 4r \int_{\partial B'_r} (\lambda_+ u^+ + \lambda_- u^-) \\ &\quad - 2(1-a) \int_{\partial B_r} |x_n|^a u \cdot u_\nu + 2r \int_{\partial B_r} |x_n|^a u_\nu^2 \end{aligned}$$

Now multiply both sides of the equation by $-r^{-n+1-a}$ to obtain that for a.e. $r \in (0, 1)$

$$\begin{aligned} 0 &= \left[\frac{1}{r^{n-a}} \int_{B_r} |x_n|^a |\nabla u|^2 \right]' + \left[\frac{4}{r^{n-a}} \int_{B'_r} (\lambda_+ u^+ + \lambda_- u^-) \right]' \\ &\quad - \frac{2}{r^{n-a}} \int_{\partial B_r} |x_n|^a \left(\frac{(1-a)u u_\nu}{r} - u_\nu^2 \right). \end{aligned}$$

As in [All12], we also have that for almost every r

$$(A.2) \quad \frac{d}{dr} \left[\frac{1-a}{r^{n+1-a}} \int_{\partial B_r} |x_n|^a u^2 \right] = \frac{2}{r^{n-a}} \int_{\partial B_r} |x_n|^a \left(\frac{(1-a)u u_\nu}{r} - \frac{(1-a)^2 u^2}{r^2} \right).$$

We then add and subtract the piece from (A.2) to obtain for almost every r

$$\begin{aligned} 0 &= \left[\frac{1}{r^{n-a}} \int_{B_r} |x_n|^a |\nabla u|^2 \right]' + \left[\frac{4}{r^{n-a}} \int_{B'_r} (\lambda_+ u^+ + \lambda_- u^-) \right]' \\ &\quad - \left[\frac{1-a}{r^{n+1-a}} \int_{\partial B_r} |x_n|^a u^2 \right]' - \frac{2}{r^{n-a}} \int_{\partial B_r} |x_n|^a \left(\frac{(1-a)u}{r} - u_\nu \right)^2 \end{aligned}$$

Thus, $W' \geq 0$, and $W' = 0$ on the interval $r_1 < r < r_2$ if and only if u is homogeneous of degree $2s = (1-a)$ on the ring $r_1 < |x| < r_2$. \square

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